The Geometry of Relativity (and maybe Gauge Theory)

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Road Map

01 Physical Motivation

02 Differential Geometry

03 Einstein’s Equation

04 Extra Topic: Gauge Theory

Part 1 | Preliminary Concepts

Euclidean Geometry, Metrics,
Special Relativity, General Relativity

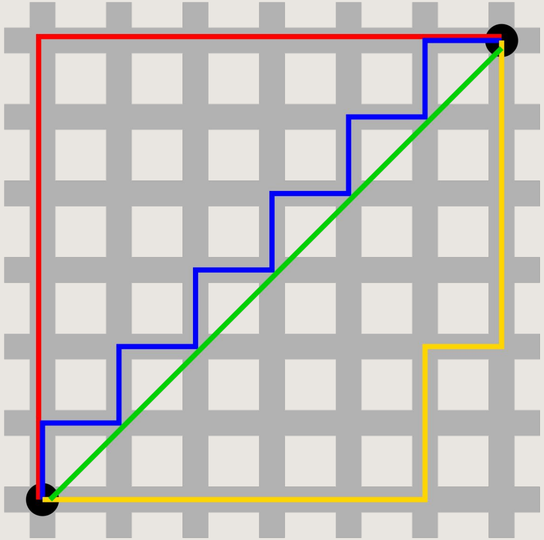
Euclidean Geometry

- What is invariant of our **choice of description** (coordinates)?
    - View through "passive" transformations

- Rotations and translations

Metrics

- Coordinates are nonetheless useful
- Metric: dictionary between coordinates and actual distances



$$\Delta \ell^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$\Delta l = |\Delta x| + |\Delta y|$$

A Brief Aside on Tensors and Index Notation

- We will view vectors as having an “upstairs” index:

$$\vec{v} \to v^{\mu}$$

- Downstairs indices represent objects that depend on one or more vectors (repeated indices are implicitly summed over):

$$\nabla_{\mu} \longrightarrow (\hat{n} \cdot \nabla) \to n^{\mu} \nabla_{\mu}$$

- Mixed indices represent objects which take in a vector, and return another vector:

$$M^{\mu}_{\nu} \longrightarrow M^{\mu}_{\nu} v^{\nu} = u^{\mu}$$

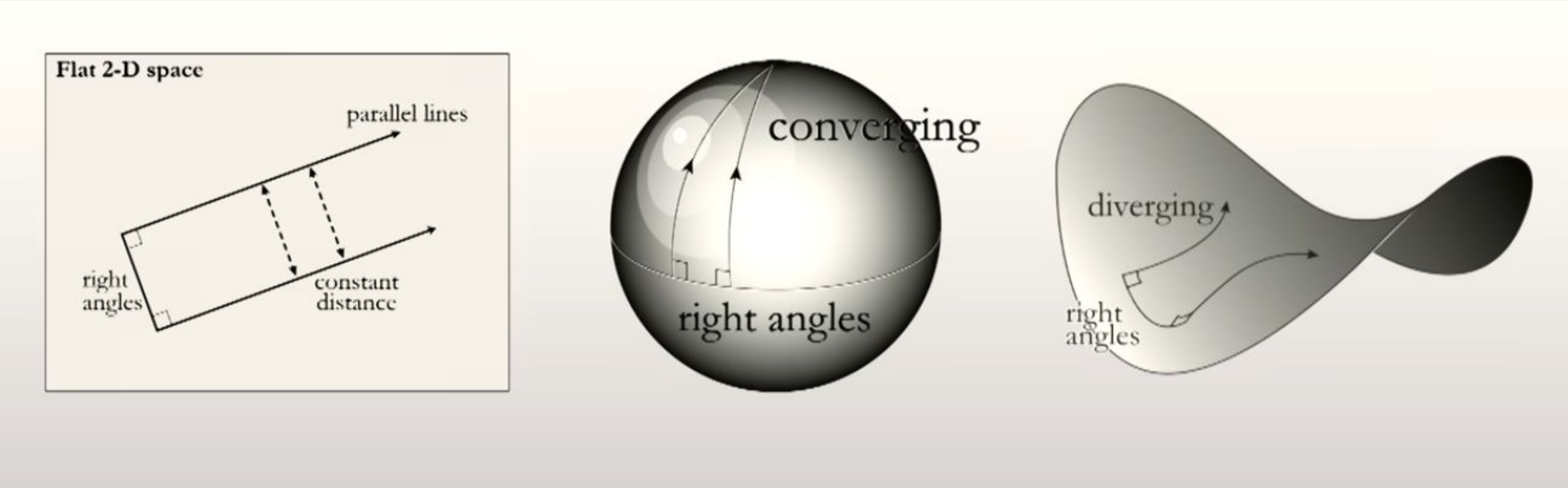
Special Relativity

- Postulates: laws of physics are invariant under...
  - Rotations
  - Spatial Translations
  - Velocity
  - Time Translations
- Electromagnetism fundamental in development, for which constancy of light *emerges*
- Distance and time intervals no longer invariant, so is everything relative?
- Merging space and time into one entity, *spacetime*, introduces new invariant *spacetime interval:*

$$\Delta s^2 = -\Delta(ct)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

Curved Space is different

- Without external forces, trajectories are ultimately still straight lines through spacetime



Gravity

Trajectory through space

$$\vec{F} = \cancel{m}\vec{a}$$

$$k\frac{qQ}{r^2} \qquad G\frac{\cancel{m}M}{r^2}$$

- Gravity isn't like the other forces…
- "Weak Equivalence Principle" has been known for centuries, tested to 10^{-15}
- Inertial frames are very important to mechanics: unique set of "neutral" frames from which to acceleration is defined with respect to
- When gravity present, impossible to have gravitationally "neutral" frame
  - "Einstein and Strong EP": In sufficiently local frames, Special Relativity holds, i.e. cannot determine whether in gravitational field

Gravity as Geometry

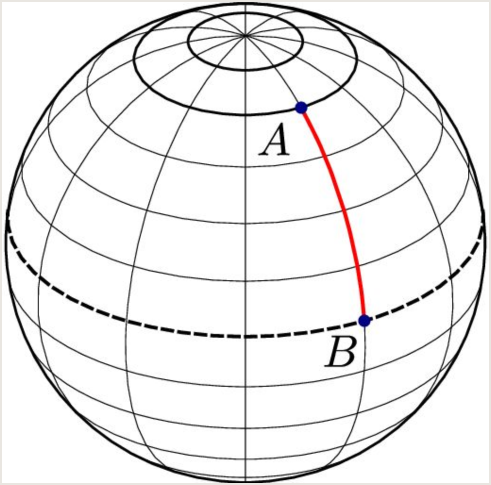
- Is impossibility of “neutral” frames a fantastic coincidence, or indicating something fundamental?
- Motion through space is just a *curve* in spacetime
- Special Relativity: Accelerated trajectories can be *embedded* within an inertial frame (flat space), from which we can reference the physics
- General Relativity: Impossible to make such a separation, gravitationally accelerated trajectories are somehow *intrinsic*
- Where are non-straight lines intrinsic? Non-Euclidean geometry! However should *locally* give special relativity

Part 2 | Differential Geometry

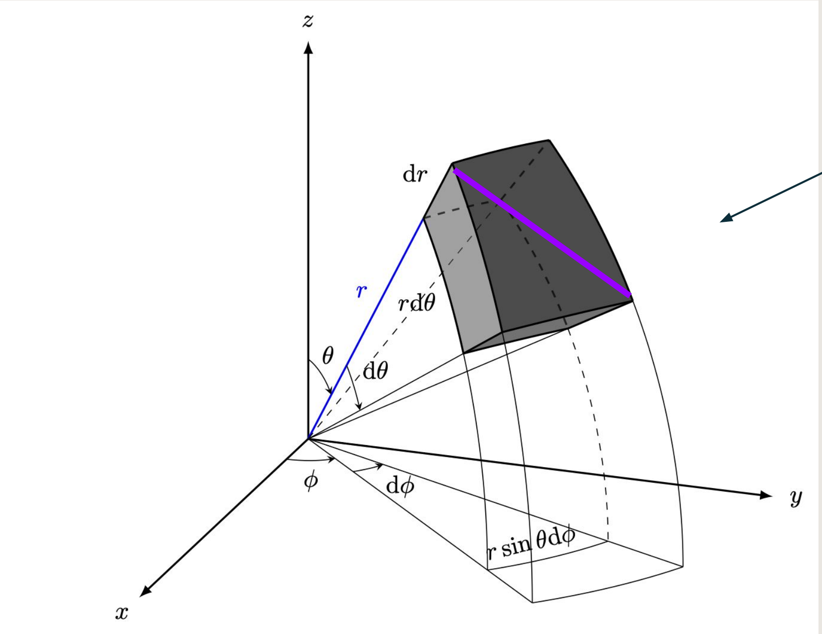
Geodesics, Manifolds,
Covariant Derivatives,
Curvature Tensors

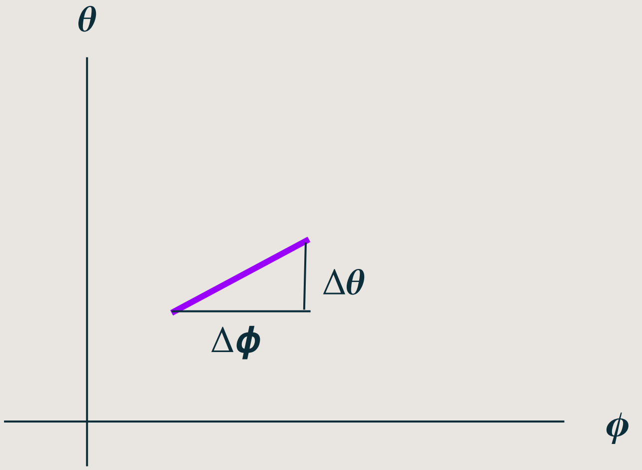
Geodesics

- Euclidean space: shortest path is straight line
- Special Relativity: extremal paths are straight lines (timelike path *maximal* length)
- Curved Space: given a distance measure, aka *metric*, shortest path is **geodesic**



Spherical Coordinates



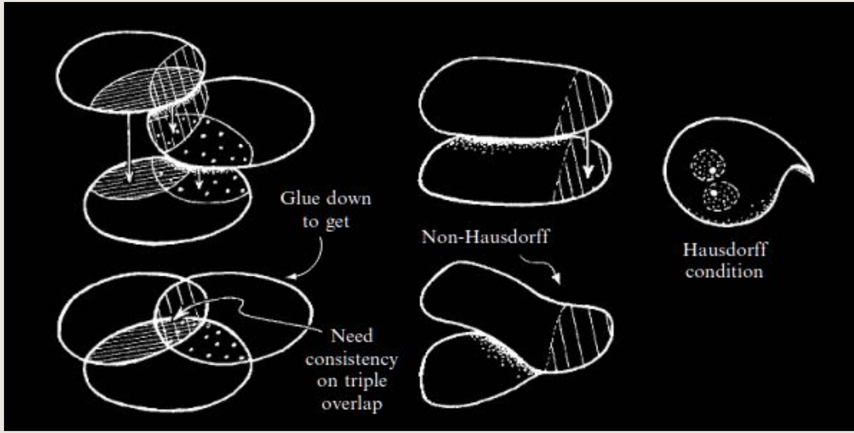


$$\cancel{\Delta\ell = \sqrt{(\Delta\theta)^2 + (\Delta\phi)^2}}$$

$$d\ell^2 = r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

- Coordinates go “bad” for θ = 0

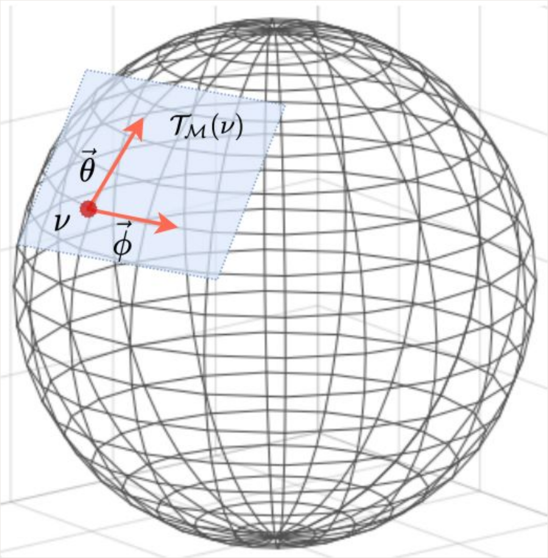
Manifolds



- **Manifold:** Our curved, well-behaved space
  - Topological: Notions of limits, connected, continuous
  - Differentiable
  - Locally Euclidean (Minkowski) space
- What type of curved spaces do we want to consider?
- Connected
- "Patchwise" coordinates: coordinates not universal, but can "patch" them together. Must agree on overlap
- Non-branching (Hausdorff)
- Orientable: Consistent sense of direction/"handedness" can be defined
  - Time orientable: consistent sense of "forward in time"

Tangent Space

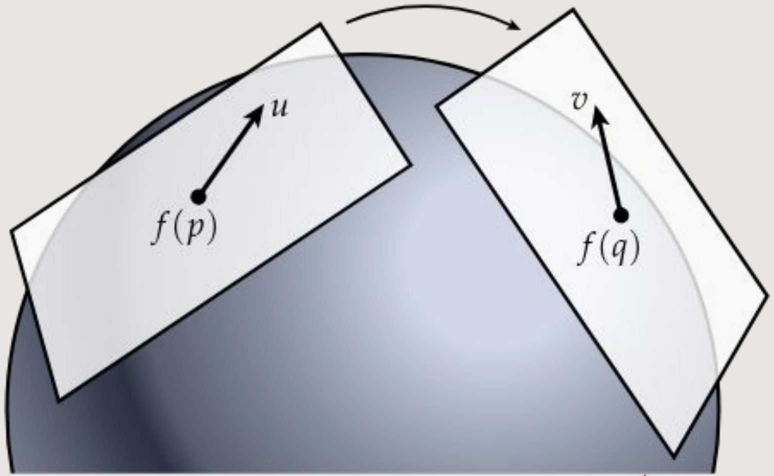
- **Tangent Space:** How to put vectors on Manifold? Local description, vector space with same dimension as manifold



- Any basis is fine (after all, it is a vector space)
- **Coordinate Basis** provides a basis which locally follows coordinate curves

Connection and Parallel Transport

- How should we compare two nearby tangent spaces? Need a notion of *connection*. We want parallel vectors to get mapped to parallel vectors, i.e. *parallel transport*



- There are an infinity of connections which provide parallel transport, however in general <u>do not preserve inner product</u>
- The unique (torsion-free) connection which does is known as the **Levi-Civita connection**

Covariant Derivative and Christoffel Symbols $\nabla_\mu \to (\hat{n} \cdot \nabla)$

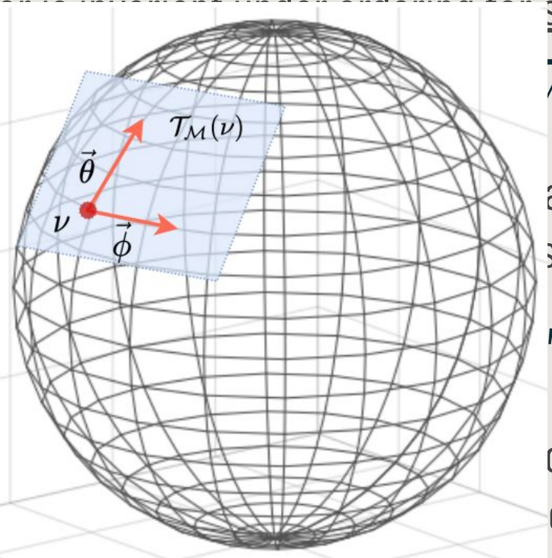
- Turns out, any *differential operator* provides a notion of connection, again with infinite number. The Levi-Civita connection we call *the* **covariant derivative**
- If a differential operat … scalar functions, we say it is *torsion-free:*

$$\ldots \nabla_\mu f$$

- The action of two diffe … aps vectors to the same vector space, just in a differe … s "matrix" to transform between

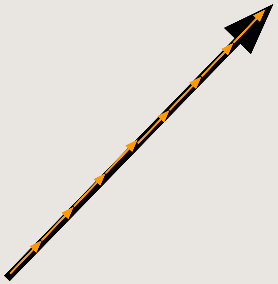
$$(\hat{n} \cdot \ldots \mathbf{T}(\vec{n})^\mu_\nu v^\nu$$

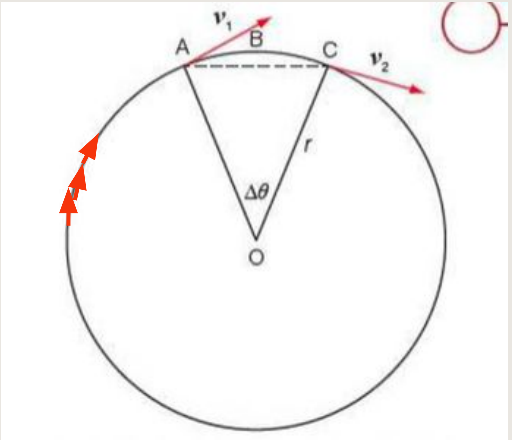
- Partial derivative (alon … operator. The **Christoffel Symbol** is object whic … oordinates

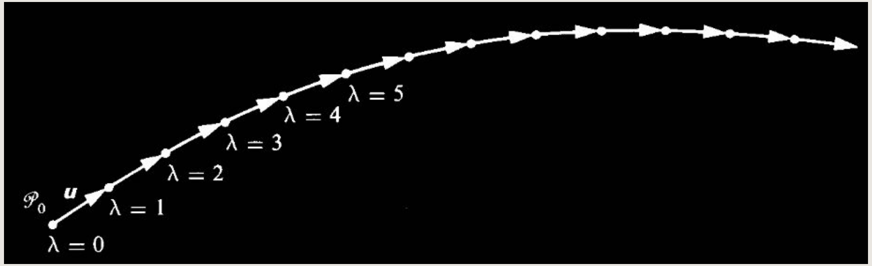


Revisiting Geodesics

- In flat space, what is the behavior of a particle exhibiting non-accelerating motion?

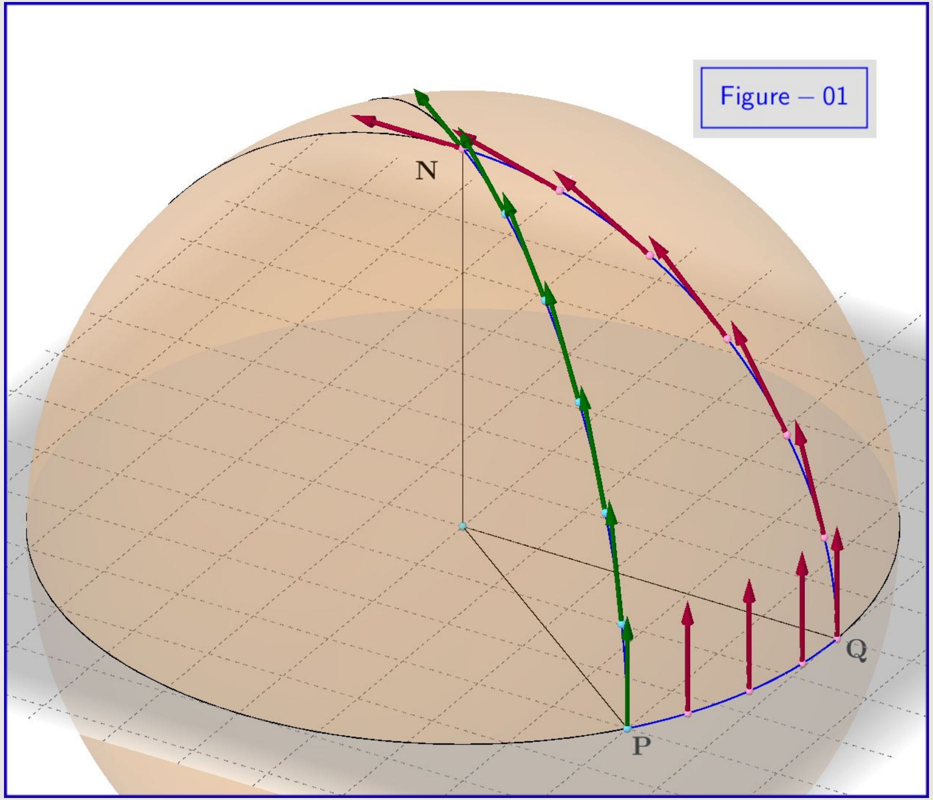






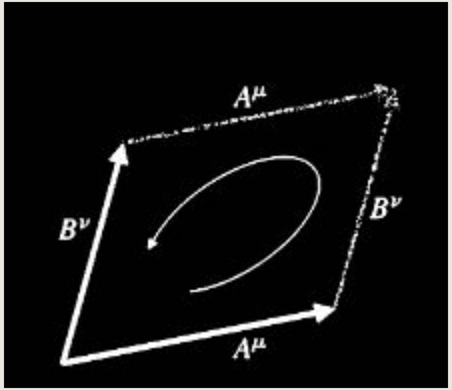
$$(\vec{v} \cdot \nabla)\vec{v} = 0$$

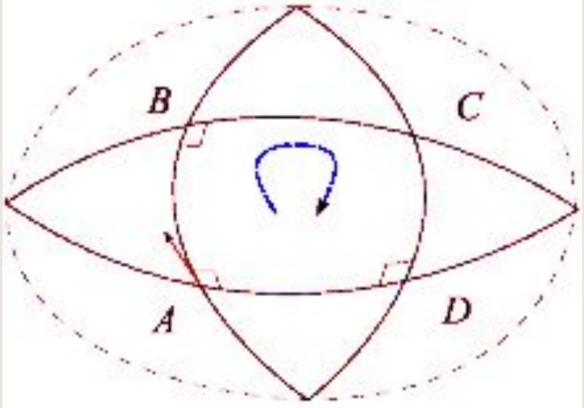
Covariant Derivatives are Path Dependent!



Curvature and the Riemann Tensor

- Flat space: everyone agrees on a global direction
- Curved space: discrepancy between local directions
- Curvature is just a measure of path dependence





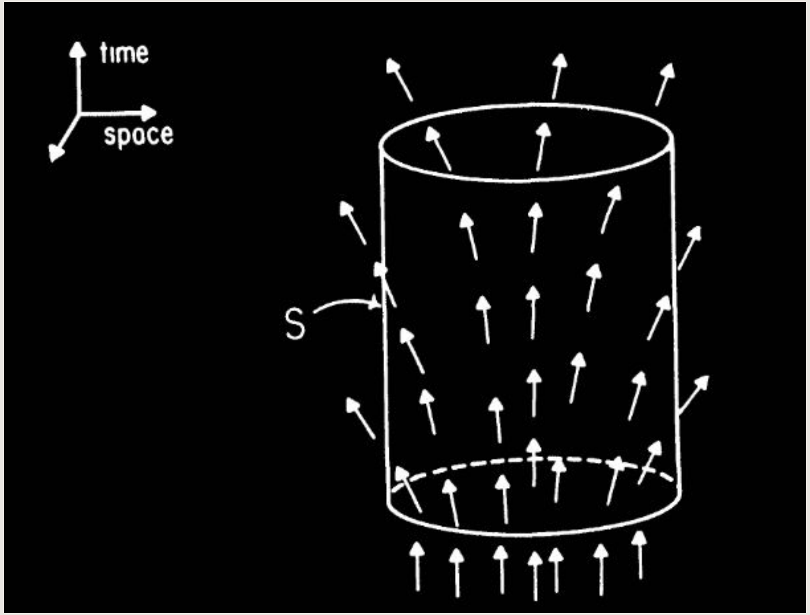
$$(\vec{A}\cdot\nabla)(\vec{B}\cdot\nabla)v^\mu - (\vec{B}\cdot\nabla)(\vec{A}\cdot\nabla)v^\mu = Riem(\vec{A},\vec{B})^\mu_{\ \nu}v^\nu$$

Part 3 | Einstein's Equation

Curvature = Matter, Energy-Momentum Tensor

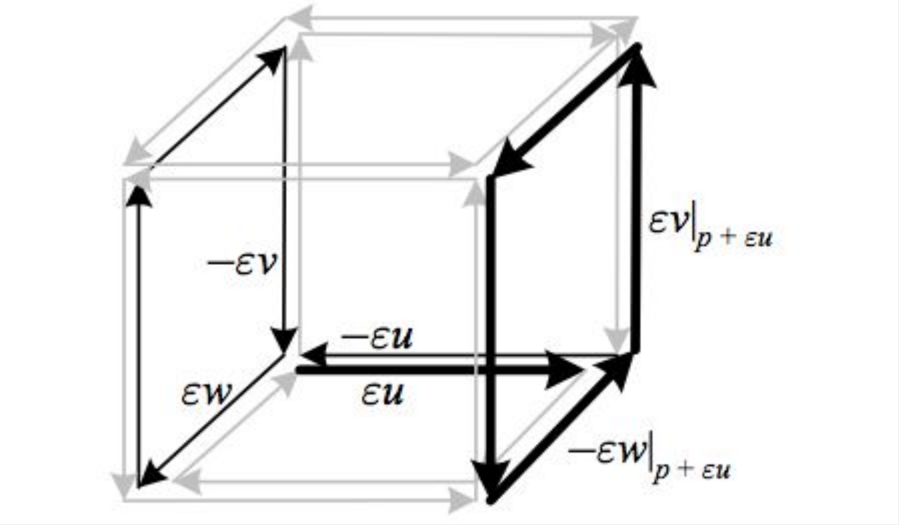
- We seek to put on solid footing our original motivation: relating local curvature and the local matter content in a region
- Consider some (at least locally) continuous matter distributions, we should have some notion of local energy-momentum conservation:
- Should remind you of Gauss's law, some divergence which is zero:

$$\nabla_\mu T^{\mu\nu} = 0$$



Bianchi's Second Identity

$$(\vec{u} \cdot \nabla)Riem(\vec{v}, \vec{w}) + (\vec{w} \cdot \nabla)Riem(\vec{u}, \vec{v}) + (\vec{v} \cdot \nabla)Riem(\vec{w}, \vec{u}) = 0$$



$$(\vec{u} \cdot \nabla)Riem(\vec{v}, \vec{w})$$

Einstein's Field Equations

- "Summing" over boundary is zero…can be recast as a divergence!
- Einstein's Field Equations is essentially a proposition: such a conserved divergence of curvature is equivalent to the conservation of energy-momentum

$$\nabla \cdot (\mathbf{Ric} - \frac{1}{2}\mathbf{g}R) \sim \nabla \cdot \mathbf{T}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Why are the Field Equations so Complicated?

$$
\begin{aligned}
&\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} + \frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\nu g_{\mu\beta} - \frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} - \frac{3}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} - \frac{1}{2} \\
&\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\rho=0}^{3}\sum_{\lambda=0}^{3} g^{\beta\lambda}g^{\alpha\rho}\partial_\alpha g_{\rho\lambda}\partial_\mu g_{\beta\nu} - \frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\rho=0}^{3}\sum_{\lambda=0}^{3} g^{\beta\lambda}g^{\alpha\rho}\partial_\alpha g_{\rho\lambda}\partial_\nu g_{\mu\beta} + \frac{1}{4}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\rho=0}^{3} \\
&\sum_{\lambda=0}^{3} g^{\beta\lambda}g^{\alpha\rho}\partial_\nu g_{\alpha\lambda}\partial_\mu g_{\rho\beta} + \frac{1}{4\left|g\right|}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\beta\left|g\right|\partial_\nu g_{\mu\alpha} - \frac{1}{4\left|g\right|}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\beta\left|g\right|\partial_\alpha g_{\mu\nu} - \frac{1}{4\left|g\right|}\sum_{\alpha=0}^{3} \\
&\sum_{\beta=0}^{3} g^{\alpha\beta}\partial_\beta\left|g\right|\partial_\mu g_{\alpha\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}
\end{aligned}
$$

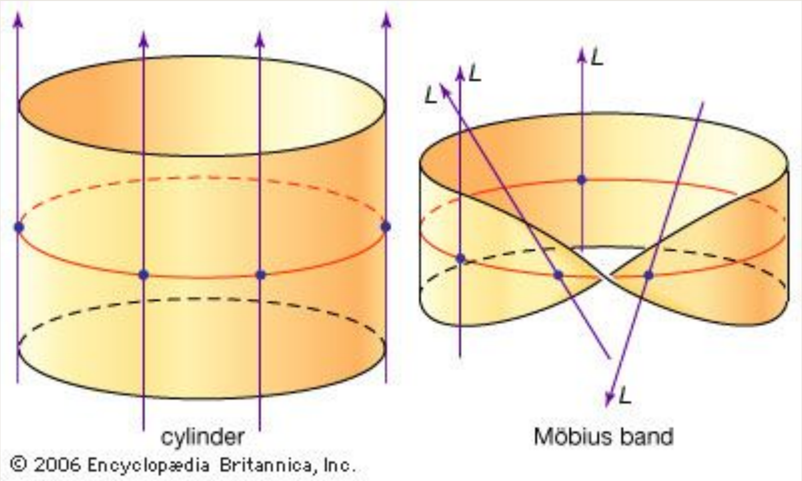
That's GR, thanks for listening!

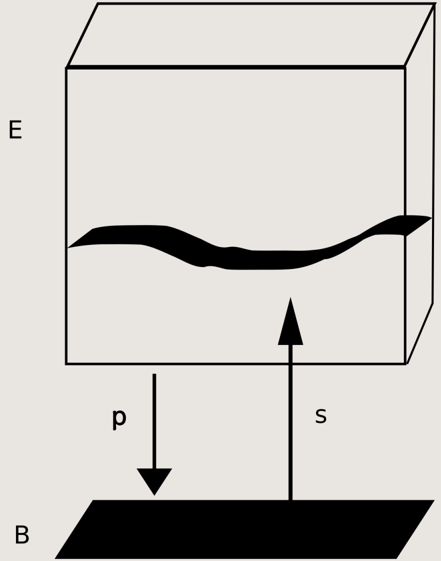
(if we have time, here's some Gauge Theory)

Part 4 | Gauge Theory

Visualizing Gauge Theory

- **Fiber:** Space of allowable values at a point
- **Fiber Bundle:** Entire set of fibers
- **Section:** Continuous path through bundle





The Gauge Theory Dictionary

- **Fiber:** analogous tangent space at a point
  - Can be a different dimension than manifold
  - Symmetries are of Gauge Group, rather than Lorentz invariance
- **Connection/Covariant Derivative:** how to related nearby fibers
- **Gauge:** A specific configuration/section through the fiber bundle

That’s actually all now, thanks!

Geometry of Special Relativity

$$x^2 + y^2 \sim const$$

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$(it)^2 + x^2$$

$$-t^2 + x^2 \sim const$$

$$\cos(i\beta) = \frac{1}{2}(e^{-\beta} + e^{\beta}) = \cosh(\beta)$$

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$